Classical stochasticity threshold and quantum mechanics

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The stochastic heating threshold for a classical model is derived by its quantized version. The model is taken from plasma physics being a particle in a constant magnetic field, which is supposed to be weak, under the effect of a sinusoidal wave. We get large fluctuations in momentum if a criterion we derive is satisfied, showing a wide indeterminacy at quantum level when compared with the initial state. It is assumed that the cyclotron frequency and the wave frequency must have an irrational ratio and that the quantum Larmor radius for the ground state is much larger than the inverse of the wave number.

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A deep question one has to face when dealing with quantum mechanics is the understanding of the mechanism, at a quantum level, of the classical chaoticity of a large class of nonlinear Hamiltonian systems. It is a well-known fact that the diffusional behavior of the classical model cannot be found at the quantum level, as could be seen for the quantum kicked rotor [1,2]. Due to the great difficulties one encounters treating with such kind of problems, some authors even suggest to modify the Schrödinger equation [2]. Then, it becomes of paramount importance to find some simple models that, in some approximation, could give an explanation of the classical behavior by standard quantum mechanics.

In this paper, consistent with the above philosophy, we show how to explain stochastic heating by discussing the widely studied model, at least at a classical level, of a harmonic oscillator; that is, a particle in a constant magnetic field taken to be weak, under the effect of a sinusoidal wave propagating orthogonally to the field. This model is interesting because, classically, it gives rise to stochastic heating when a certain threshold of the wave-field intensity is overcome, both for small or large magnetic fields [3]. Our study consists of the analysis of the quantum behavior of such a model that, under certain conditions, permits the derivation of the classical stochastic heating threshold.

In order to fix the limitations in our description and to have the notation defined, we write down the Hamiltonian of the model as [4]

$$H=(a^+a+\tfrac{1}{2})\hbar\omega_0+q\phi\cos[\beta(a^++a)-\omega t] \tag{1}$$

a and a^+ being the ladder operators, ω_0 the cyclotron frequency, q the charge of the particle, ϕ the potential, and $\beta=(\hbar k^2/2m\omega_0)^{1/2}=kx_0$, with k the wave number. This is the quantized version of the model discussed in Ref. [3]. As pointed out in Refs. [4], our results hold with the condition that $\beta\gg1$, similar to the classical condition on the Larmor radius. In addition, we assume that the ratio between the cyclotron frequency and the wave frequency must be an irrational number, typical of the Kolmogorov-Arnold-Maser theorem [1,3].

As shown with the method used in Refs. [4], the wave function for our model can be written as

$$|\psi(t)\rangle\approx\sum_{n=-\infty}^{+\infty}J_n(z)e^{-in\omega t}|in\beta\rangle \tag{2}$$

where $z=q\phi/\hbar\omega$, J_n is the nth Bessel function, and $|in\beta\rangle$ a coherent state having $in\beta$ as a complex parameter. Equation (2) is defined as a phase factor due to the ground state energy [4].

Equation (2) holds, given the above limitations, with the initial condition that in the far past it coincides with the ground state of the harmonic oscillator. With Eq. (2) we can immediately compute the mean values of the position and momentum, and their higher powers. For the position $x=x_0(a^++a)$, in the limit $\beta\gg1$, we obtain

$$\langle x\rangle=ix_0\beta\sum_{\substack{m,n\\m\neq n}}(n-m)J_m(z)J_n(z)e^{i(m-n)\omega t}e^{-(m-n)^2(\beta^2/2)}\approx0\,, \tag{3}$$

$$\langle x^2\rangle=x_0^2+x_0^2\sum_{\substack{m,n\\m\neq n}}[1-(m-n)^2\beta^2]J_m(z)J_n(z)\times e^{i(m-n)\omega t}e^{-(m-n)^2(\beta^2/2)}\approx x_0^2 \tag{4}$$

$$\langle x^3\rangle=ix_0^3\sum_{\substack{m,n\\m\neq n}}[(m-n)^3\beta^3-3(m-n)\beta]J_m(z)J_n(z)\times e^{i(m-n)\omega t}e^{-(m-n)^2(\beta^2/2)}\approx0 \tag{5}$$

$$\langle x^4\rangle=3x_0^4+x_0^4\sum_{\substack{m,n\\m\neq n}}\{(m-n)^4\beta^4+3[1-2(m-n)^2\beta^2]\}\times J_m(z)J_n(z)e^{i(m-n)\omega t}e^{-(m-n)^2(\beta^2/2)}\approx3x_0^4\,. \tag{6}$$

We easily realize that we have the early expectation values for the position and its powers as computed by the ground state wave function of the harmonic oscillator. The particle approximately keeps its localization on the basis of this result. Things can change radically for the momentum, $p=ip_0(a^+-a)$, where $p_0=(m\hbar\omega_0/2)^{1/2}$. In fact, a calculation similar to the one above for the momentum gives

$$\langle p \rangle = p_0\beta \sum_{\substack{m,n\\ m\neq n}} (m+n)J_m(z)J_n(z)e^{i(m-n)\omega t}e^{-(m-n)^2(\beta^2/2)}$$
$$\approx 0 , \quad (7)$$

$$\langle p^2 \rangle = p_0^2(1+2\beta^2z^2)+p_0^2 \sum_{\substack{m,n\\ m\neq n}} [1+(m+n)^2\beta^2]J_m(z)J_n(z)e^{i(m-n)\omega t}e^{-(m-n)^2(\beta^2/2)}$$
$$\approx p_0^2(1+2\beta^2z^2) , \quad (8)$$

$$\langle p^3 \rangle = p_0^3 \sum_{\substack{m,n\\ m\neq n}} [(m+n)^3\beta^3+3(m+n)\beta]J_m(z)J_n(z)e^{i(m-n)\omega t}e^{-(m-n)^2(\beta^2/2)}$$
$$\approx 0 , \quad (9)$$

$$\langle p^4 \rangle = p_0^4(3+6\beta^4z^4+8\beta^4z^2+12\beta^2z^2)+p_0^4 \sum_{\substack{m,n\\ m\neq n}} \{(m+n)^4\beta^4+3[1+2(m+n)^2\beta^2]\}J_m(z)J_n(z)e^{i(m-n)\omega t}e^{-(m-n)^2(\beta^2/2)}$$
$$\approx p_0^4(3+6\beta^4z^4+8\beta^4z^2+12\beta^2z^2) . \quad (10)$$

We easily realize that now the probability distribution for the momentum is quite different from the initial Gaussian, depending on the critical parameter $\beta|z|$. The indeterminacy for the momentum can become very large, in this way increasing the number of cells of phase space accessible to the particle that can become strongly delocalized in the momentum, while keeping its space localization. In order for this to happen, we have to require that $\Delta p/p_0 \approx \sqrt{2}\beta|z| \gg 1$. This takes us directly to

$$2\beta^2|z| = \frac{|q|k^2\phi}{m\omega\omega_0} \gg 1 , \quad (11)$$

making our result independent of $\hbar$. This is just the criterion for stochastic heating in the case of a weak magnetic field [3], generally derived with the Chirikov overlap criterion [1,3].

In conclusion, we have seen that the method and model of Ref. [4] give a possible quantum explanation of classical stochastic heating. We remark simply that the explanation of classical chaos by standard quantum mechanics could come from the development of good methods and concepts and not from modifying the theory itself.

[1] L. E. Reichl, in *In Conservative Classical Systems: Quantum Manifestations* (Springer-Verlag, New York, 1992). The quantum kicked rotor is discussed in Chap. 9, p. 407ff.

[2] See, e.g., K. Nakamura, *Quantum Chaos* (Cambridge University Press, Cambridge, 1993).

[3] R. Z. Sagdeev, D. A. Usikov, and G. M. Zaslavsky, in *From the Pendulum to Turbulence and Chaos* (Harwood Academic, Philadelphia, 1992), and the reference therein in Chap. 13. The classical stochasticity threshold for the model considered in this report is given in note 6 of Chap. 13.

[4] M. Frasca, Nuovo Cimento **107B**, 845 (1992); **109B**, 1227 (1994).